

Expansion of the Yang-Mills Hamiltonian in spatial derivatives and glueball spectrum

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Abstract

A strong coupling expansion of the $SU(2)$ Yang-Mills quantum Hamiltonian is carried out in the form of an expansion in the number of spatial derivatives, using the symmetric gauge $\epsilon_{ijk}A_{jk} = 0$. Introducing an infinite lattice with box length a , I obtain a systematic strong coupling expansion of the Hamiltonian in $\lambda \equiv g^{-2/3}$, with the free part being the sum of Hamiltonians of Yang-Mills quantum mechanics of constant fields for each box, and interaction terms of higher and higher number of spatial derivatives connecting different boxes. The corresponding deviation from the free glueball spectrum, obtained earlier for the case of the Yang-Mills quantum mechanics of spatially constant fields, is calculated using perturbation theory in λ . As a first step, the interacting glueball vacuum and the energy spectrum of the interacting spin-0 glueball are obtained to order λ^2 . Its relation to the renormalisation of the coupling constant in the IR is discussed, indicating the absence of infrared fixed points.

1 Introduction

A very promising method for non-perturbative investigations of Yang-Mills theory has turned out to be the Hamiltonian approach [1], in particular the possibility to use the powerful variational method.

I shall consider here the Yang-Mills theory of $SU(2)$ gauge fields $A_\mu^a(x)$, defined by the action

$$\mathcal{S}[A] := -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} , \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c , \quad (1)$$

invariant under both Poincaré and scale transformations, and the local $SU(2)$ gauge transformations $U[\omega(x)] \equiv \exp(i\omega_a \tau_a/2)$

$$A_{a\mu}^\omega(x) \tau_a/2 = U[\omega(x)] \left(A_{a\mu}(x) \tau_a/2 + \frac{i}{g} \partial_\mu \right) U^{-1}[\omega(x)] . \quad (2)$$

The transition to the corresponding quantum theory is then carried out by exploiting the time dependence of the gauge transformations to put

$$A_{a0}(x) = 0 , \quad a = 1, 2, 3 , \quad (\text{Weyl gauge})$$

and impose canonical commutation relations on the spatial fields using the Schrödinger representation

$$[\Pi_{ai}(\mathbf{x}), A_{bj}(\mathbf{y})] = i\delta_{ab}\delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \quad \longrightarrow \quad \Pi_{ai}(\mathbf{x}) = -E_{ai}(\mathbf{x}) = -i\delta/\delta A_{ai}(\mathbf{x}) .$$

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The physical states Ψ have to satisfy the system of equations

$$\begin{aligned} (H - E)\Psi &= 0 \quad (\text{Schrödinger equation}) , \\ G_a(\mathbf{x})\Psi &= 0 \quad (\text{Gauss law constraints}) , \end{aligned} \quad (3)$$

with the Hamiltonian ($B_{ai}(A) = \epsilon_{ijk} \left(\partial_j A_{ak} + \frac{1}{2} g \epsilon_{abc} A_{bj} A_{ck} \right)$)

$$H = \int d^3\mathbf{x} \frac{1}{2} \sum_{a,i} \left[\left(\frac{\delta}{\delta A_{ai}(\mathbf{x})} \right)^2 + B_{ai}^2(A(\mathbf{x})) \right] , \quad (4)$$

and the Gauss law operators

$$G_a(\mathbf{x}) = -i (\delta_{ac} \partial_i + g \epsilon_{abc} A_{bi}(\mathbf{x})) \frac{\delta}{\delta A_{ci}(\mathbf{x})} , \quad a = 1, 2, 3 , \quad (5)$$

which are the generators of the residual time-independent gauge transformations, commute with the Hamiltonian and satisfy angular momentum commutation relations

$$[G_a(\mathbf{x}), H] = 0 , \quad [G_a(\mathbf{x}), G_b(\mathbf{y})] = i g \epsilon_{abc} G_c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) ,$$

The matrix elements are

$$\langle \Phi_1 | \mathcal{O} | \Phi_2 \rangle = \int \prod_{ik} dA_{ik} \Phi_1^* \mathcal{O} \Phi_2 . \quad (6)$$

2 Physical $SU(2)$ Quantum Hamiltonian in the symmetric gauge

In order to calculate the eigenstates and their energies, it is useful to implement the non-Abelian Gauss law constraints into the Schrödinger equation by further fixing the gauge using the remaining time-independent gauge transformations. One possibility, well suited for the high energy sector of the theory, is to impose the Coulomb gauge $\chi_a(A) = \partial_i A_{ai} = 0$ describing the dynamics in terms of physical colored transverse gluons. I shall here choose the symmetric gauge[2]¹

$$\chi_i(A) = \epsilon_{ijk} A_{jk} = 0 \quad (\text{symmetric gauge}) . \quad (7)$$

In contrast to the Coulomb gauge, the symmetric gauge allows for an expansion of the physical Hamiltonian in spatial derivatives, which makes it very suited for the study of the infrared sector of Yang-Mills theory. The physical degrees of freedom in the symmetric gauge are the six components of a colorless local symmetric tensor field.

The symmetric gauge corresponds to the point transformation to the new set of adapted coordinates, the three q_j ($j = 1, 2, 3$) and the six elements $S_{ik} = S_{ki}$ ($i, k = 1, 2, 3$) of the positive definite² symmetric 3×3 matrix S

$$A_{ai}(q, S) = O_{ak}(q) S_{ki} - \frac{1}{2g} \epsilon_{abc} \left(O(q) \partial_i O^T(q) \right)_{bc} , \quad (8)$$

where $O(q)$ is an orthogonal 3×3 matrix parametrized by the q_i . After the above coordinate transformation (8), the non-Abelian Gauss law constraints become the Abelian conditions

$$G_a \Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i} \Phi = 0 \quad (\text{Abelianisation}),$$

that the physical states should depend only on the physical variables S_{ik} , and the system (3) reduces to the unconstrained Schrödinger equation

$$H(S, P) \Phi(S) = E \Phi(S) . \quad (9)$$

¹It has been proven in [3], that the symmetric gauge exists (at least for strong coupling), by showing that any time-independent gauge field can be carried over uniquely into the symmetric gauge.

²In the infinite coupling limit this transformation reduces to the polar decomposition, in which the symmetric matrix can be chosen to be positive definite.

The correctly ordered physical quantum Hamiltonian [1] in the symmetric gauge in terms of the physical variables $S_{ik}(\mathbf{x})$ and the corresponding canonically conjugate momenta $P_{ik}(\mathbf{x}) \equiv -i\delta/\delta S_{ik}(\mathbf{x})$ reads

$$H(S, P) = \frac{1}{2} \mathcal{J}^{-1} \int d^3\mathbf{x} \int d^3\mathbf{y} P_{mn}(\mathbf{x}) \mathcal{J} \mathcal{K}_{mn|st}(\mathbf{x}, \mathbf{y}) P_{st}(\mathbf{y}) + \frac{1}{2} \int d^3\mathbf{x} (B_{ai}(S))^2 , \quad (10)$$

with the kernel

$$\mathcal{K}_{mn|st}(\mathbf{x}, \mathbf{y}) := \delta_{ms} \delta_{nt} \delta(\mathbf{x} - \mathbf{y}) - 2 \langle \mathbf{x} \ n | D_m(S) {}^*D^{-2}(S) D_s(S) | \mathbf{y} \ t \rangle , \quad (11)$$

the Jacobian

$$\mathcal{J} \equiv \det |{}^*D| , \quad (12)$$

the covariant derivative

$$D_i(S)_{kl} \equiv \delta_{kl} \partial_i - g \epsilon_{klm} S_{mi} ,$$

the Faddeev-Popov (FP) operator

$${}^*D_{kl}(S) \equiv \epsilon_{kmi} D_i(S)_{ml} = \epsilon_{kli} \partial_i - g \gamma_{kl}(S) , \quad \gamma_{kl}(S) \equiv S_{kl} - \delta_{kl} \text{tr} S , \quad (13)$$

and the Green function

$$\langle \mathbf{x} \ a | {}^*D^{-2}(S) | \mathbf{y} \ b \rangle \equiv {}^*D_{ab}^{-2}(S)^{(\mathbf{x})} [\delta(\mathbf{x} - \mathbf{y})] .$$

The matrix element of a physical operator O is given by

$$\langle \Psi' | O | \Psi \rangle \propto \int \prod_{\mathbf{x}} [dS(\mathbf{x})] \mathcal{J} \Psi'^*[S] O \Psi[S] . \quad (14)$$

A great advantage of the symmetric gauge - in contrast for example to the Coulomb gauge-, is that the corresponding FP operator, and hence the non-local terms of the physical Hamiltonian, can be expanded in the number of spatial derivatives. The Green function

$$\langle \mathbf{x} \ k | {}^*D^{-1}(S) | \mathbf{y} \ l \rangle \equiv {}^*D_{kl}^{-1}(S)^{(\mathbf{x})} [\delta(\mathbf{x} - \mathbf{y})] ,$$

corresponding to the FP operator (13), can be expanded in the number of spatial derivatives

$$\begin{aligned} \langle \mathbf{x} \ k | {}^*D^{-1}(S) | \mathbf{y} \ l \rangle &= -\frac{1}{g} \gamma_{kl}^{-1}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) + \frac{1}{g^2} \gamma_{ka}^{-1}(\mathbf{x}) \epsilon_{abc} \partial_c^{(\mathbf{x})} \left[\gamma_{bl}^{-1}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \right] \\ &\quad - \frac{1}{g^3} \gamma_{ka}^{-1}(\mathbf{x}) \epsilon_{abc} \partial_c^{(\mathbf{x})} \left[\gamma_{bi}^{-1}(\mathbf{x}) \epsilon_{ijs} \partial_s^{(\mathbf{x})} \left[\gamma_{jl}^{-1}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \right] \right] + \dots . \end{aligned}$$

3 Expansion of the Hamiltonian in spatial derivatives

In order to perform a consistent expansion of the physical Hamiltonian in spatial derivatives, also the non-locality in the Jacobian \mathcal{J} has to be taken into account. This will be achieved in the following way.

Writing the FP operator in the form

$${}^*D_{kl}(S) \equiv -g \gamma_{km}(S) \left[\delta_{ml} - \frac{1}{g} \gamma_{mn}^{-1}(S) \epsilon_{nli} \partial_i \right] \equiv -g \gamma_{km}(S) {}^*\tilde{D}_{ml}(S) ,$$

the Jacobian \mathcal{J} factorizes

$$\mathcal{J} = \mathcal{J}_0 \tilde{\mathcal{J}} , \quad (15)$$

with the local

$$\mathcal{J}_0 \equiv \det |\gamma| = \prod_{\mathbf{x}} \det |\gamma(\mathbf{x})| , \quad \det |\gamma(\mathbf{x})| = \prod_{i < j} (\phi_i(\mathbf{x}) + \phi_j(\mathbf{x})) , \quad (\phi_i = \text{eigenvalues of } S) \quad (16)$$

and the non-local $\tilde{\mathcal{J}} \equiv \det |^*\tilde{D}|$. Now I include the non-local part of the measure into the wave functional

$$\tilde{\Psi}(S) := \tilde{\mathcal{J}}^{-1/2} \Psi(S) ,$$

leading to the corresponding transformed Hamiltonian $\tilde{H} := \tilde{\mathcal{J}}^{1/2} H \tilde{\mathcal{J}}^{-1/2}$, being Hermitean with respect to the local measure \mathcal{J}_0

$$\tilde{H}(S, P) = \frac{1}{2} \mathcal{J}_0^{-1} \int d^3 \mathbf{x} \int d^3 \mathbf{y} P_{mn}(\mathbf{x}) \mathcal{J}_0 \mathcal{K}_{mn|st}(\mathbf{x}, \mathbf{y}) P_{st}(\mathbf{y}) + \frac{1}{2} \int d^3 \mathbf{x} (B_{ai}(S))^2 + V_{\text{meas}}(S) , \quad (17)$$

on the cost of extra terms ³ V_{meas} from the non-local factor $\tilde{\mathcal{J}}$ of the original measure \mathcal{J}

$$V_{\text{meas}}(S) = \frac{1}{4} \mathcal{J}_0^{-1} \int d^3 \mathbf{x} \frac{\delta}{\delta S_{mn}(\mathbf{x})} \left[\mathcal{J}_0 \int d^3 \mathbf{y} \mathcal{K}_{mn|st}(\mathbf{x}, \mathbf{y}) \tilde{\Delta}_{st}(\mathbf{y}) \right] + \frac{1}{8} \int d^3 \mathbf{x} \int d^3 \mathbf{y} \tilde{\Delta}_{mn}(\mathbf{x}) \mathcal{K}_{mn|st}(\mathbf{x}, \mathbf{y}) \tilde{\Delta}_{st}(\mathbf{y}) , \quad (18)$$

with the original kernel \mathcal{K} in (11) and

$$\tilde{\Delta}_{mn}(\mathbf{x}) := \frac{\delta \ln \tilde{\mathcal{J}}}{\delta S_{mn}(\mathbf{x})} = -g \left(\langle \mathbf{x} m | ^* D^{-1} | \mathbf{x} n \rangle - \delta_{mn} \langle \mathbf{x} k | ^* D^{-1} | \mathbf{x} k \rangle \right) - \delta(\mathbf{0}) \left(\gamma_{mn}^{-1}(\mathbf{x}) - \delta_{mn} \text{tr} \gamma^{-1}(\mathbf{x}) \right) . \quad (19)$$

The matrix element (14) of a physical operator O becomes the product of local matrix elements

$$\langle \Psi' | O | \Psi \rangle \propto \int \prod_{\mathbf{x}} \left[dS(\mathbf{x}) \prod_{i < j} (\phi_i(\mathbf{x}) + \phi_j(\mathbf{x})) \right] \Psi'^*[S] O \Psi[S] . \quad (20)$$

The transformed physical Hamiltonian (17) can be expanded in the number of spatial derivatives

$$\tilde{H} = H_0 + \sum_{\alpha} V_{\alpha}^{(\partial)} + \left(\sum_{\beta} V_{\beta}^{(\Delta)} + \sum_{\gamma} V_{\gamma}^{(\partial \partial \neq \Delta)} \right) + \dots , \quad (21)$$

with the free part H_0 containing no spatial derivatives, the interaction parts $V_{\alpha}^{(\partial)}$ containing one spatial derivative, and $V_{\beta}^{(\Delta)}, V_{\gamma}^{(\partial \partial \neq \Delta)}$ containing two spatial derivatives, and so on.

3.1 The free part H_0

The free part H_0 containing no spatial derivatives reads

$$H_0 = \int d^3 \mathbf{x} \frac{1}{2} \left[(P_{mn})^2 - i \delta(\mathbf{0}) \left[\gamma_{mn}^{-1}(S) - \delta_{mn} \text{tr}(\gamma^{-1}(S)) \right] P_{mn} + \frac{1}{2} \gamma_{mn}^{-2} \mathcal{S}_m^{\text{spin}} \mathcal{S}_n^{\text{spin}} + \frac{g^2}{2} (\text{tr}^2 S^2 - \text{tr} S^4) \right] , \quad (22)$$

with the spin densities $\mathcal{S}_i^{\text{spin}} = 2 \epsilon_{ijk} S_{ja} P_{ak}$, $i = 1, 2, 3$ (note the factor 2).

In order to achieve a more transparent form for the reduced Yang-Mills system I shall limit myself in this work to the principle orbit configurations

$$0 < \phi_1 < \phi_2 < \phi_3 < \infty , \quad (23)$$

for the eigenvalues $\phi_1, \phi_2, \phi_3 > 0$ of the positive definite symmetric matrix S (not considering singular orbits where two or more eigenvalues coincide) and perform a principal-axes transformation

$$S = R(\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R^T(\alpha, \beta, \gamma) , \quad (24)$$

with the $SO(3)$ matrix R parametrized by the three Euler angles $\chi \equiv (\alpha, \beta, \gamma)$. The Jacobian of (24) is $|\partial S / \partial(\alpha, \beta, \gamma, \phi)| \propto \sin \beta \prod_{i < j} (\phi_i - \phi_j)$. The original physical variables can then be written in terms of the new canonical variables as (using Clebsch-Gordan coefficients)⁴

$$S_{ik} = C_{1i \ 1k}^{2A} (\phi_3)_A^{(2)} + \frac{1}{\sqrt{3}} \delta_{ik} (\phi_3)^{(0)} , \quad P_{ik} = C_{1i \ 1k}^{2A} (\pi_3, \xi_3)_A^{(2)} + \frac{1}{\sqrt{3}} \delta_{ik} (\pi_3)^{(0)} \quad (25)$$

³Although in principle, V_{meas} is part of the electric term of the Hamiltonian, I shall treat it separately in this work as "measure term".

⁴For spin-1 fields $S_i^{(1)}$ I use the Cartesian combinations $S_{1-}^{(1)} \equiv S_{1-}^{(1)} := (S_{+1}^{(1)} - S_{-1}^{(1)})/\sqrt{2} \equiv e_1^{\alpha} S_{\alpha}^{(1)}$, $S_2^{(1)} \equiv S_{1+}^{(1)} := i(S_{+1}^{(1)} + S_{-1}^{(1)})/\sqrt{2} \equiv e_2^{\alpha} S_{\alpha}^{(1)}$, $S_3^{(1)} \equiv S_0^{(1)} \equiv e_3^{\alpha} S_{\alpha}^{(1)}$, such that e.g. $C_{1i \ 1k}^{2A} := e_i^{\alpha} e_k^{\beta} C_{1\alpha \ 1\beta}^{2A}$. For spin-2 fields $S_A^{(2)}$ I use correspondingly the real combinations $S_{2+}^{(2)} := (S_{+2}^{(2)} + S_{-2}^{(2)})/\sqrt{2}$, $S_{2-}^{(2)} := -i(S_{+2}^{(2)} - S_{-2}^{(2)})/\sqrt{2}$, $S_{1+}^{(2)} := i(S_{+1}^{(2)} + S_{-1}^{(2)})/\sqrt{2}$, $S_{1-}^{(2)} := -(S_{+1}^{(2)} - S_{-1}^{(2)})/\sqrt{2}$.

with the spin-0 and spin-2 fields (using Wigner D-functions)

$$(\phi_3)^{(0)} := (\phi_1 + \phi_2 + \phi_3) / \sqrt{3} \quad (26)$$

$$(\phi_3)_A^{(2)} := \sqrt{\frac{2}{3}} \left[\left(\phi_3 - \frac{1}{2} (\phi_1 + \phi_2) \right) D_{A0}^{(2)}(\chi) + \frac{\sqrt{3}}{2} (\phi_1 - \phi_2) D_{A2+}^{(2)}(\chi) \right], \quad (27)$$

and (using $\pi_i \equiv -i\delta/\delta\phi_i$)

$$(\pi_3)^{(0)} := (\pi_1 + \pi_2 + \pi_3) / \sqrt{3} \quad (28)$$

$$\begin{aligned} (\pi_3, \xi_3)_A^{(2)} &:= \sqrt{\frac{2}{3}} \left[\left(\pi_3 - \frac{1}{2} (\pi_1 + \pi_2) \right) D_{A0}^{(2)}(\chi) + \frac{\sqrt{3}}{2} (\pi_1 - \pi_2) D_{A2+}^{(2)}(\chi) \right] \\ &+ \frac{1}{\sqrt{2}} \left[D_{A1+}^{(2)}(\chi) \frac{\xi_1}{\phi_2 - \phi_3} + D_{A1-}^{(2)}(\chi) \frac{\xi_2}{\phi_3 - \phi_1} + D_{A2-}^{(2)}(\chi) \frac{\xi_3}{\phi_1 - \phi_2} \right], \end{aligned} \quad (29)$$

with the intrinsic spin angular momentum densities $\xi_i(\mathbf{x}) \equiv -R_{ij}^T(\chi(\mathbf{x})) \mathcal{S}_j^{\text{spin}}(\mathbf{x})$,

$$[\mathcal{S}_i^{\text{spin}}(\mathbf{x}), \xi_j(\mathbf{y})] = 0, \quad [\xi_i(\mathbf{x}), \xi_j(\mathbf{y})] = -i\epsilon_{ijk} \delta^3(\mathbf{x} - \mathbf{y}) \xi_k(\mathbf{x}).$$

The spin vectors $\mathcal{S}_k^{\text{spin}}$, finally, can be written as

$$\mathcal{S}_k^{\text{spin}} = D_{k1-}^{(1)}(\chi) \xi_1 + D_{k1+}^{(1)}(\chi) \xi_2 + D_{k0}^{(1)}(\chi) \xi_3.$$

Hence, in terms of the principal-axes variables, the part H_0 of the physical Hamiltonian, containing no spatial derivatives, reads

$$H_0 = \int d^3\mathbf{x} \frac{1}{2} \sum_{i,j,k}^{\text{cyclic}} \left[\pi_i^2 - \frac{2i\delta(\mathbf{0})}{\phi_j^2 - \phi_k^2} (\phi_j \pi_j - \phi_k \pi_k) + \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + g^2 \phi_j^2 \phi_k^2 \right]. \quad (30)$$

The matrix elements of a physical operator O are given as

$$\langle \Psi' | O | \Psi \rangle \propto \prod_{\mathbf{x}} \int d\alpha(\mathbf{x}) \sin \beta d\beta(\mathbf{x}) d\gamma(\mathbf{x}) \int_{0 < \phi_1(\mathbf{x}) < \phi_2(\mathbf{x}) < \phi_3(\mathbf{x})} \left[\prod_{i,j,k}^{\text{cyclic}} d\phi_i(\mathbf{x}) (\phi_j^2(\mathbf{x}) - \phi_k^2(\mathbf{x})) \right] \Psi'^* O \Psi. \quad (31)$$

3.2 First and second order interaction terms

The interaction parts of first and second order in the number of spatial derivatives, needed in this work, can be written in the general form ($\Delta \equiv \partial_x^2 + \partial_y^2 + \partial_z^2$)

$$V_\alpha^{(\partial)} \equiv C_{S_1 M_1}^{1k}{}_{S_2 M_2} \int d^3x \tilde{Y}_{\alpha M_1}^{(S_1)}[\phi] i\partial_k Y_{\alpha M_2}^{(S_2)}[\phi], \quad (32)$$

$$V_\beta^{(\Delta)} \equiv - \int d^3x \tilde{X}_{\beta M}^{(S)}[\phi] \Delta X_{\beta M}^{(S)}[\phi] = V_\beta^{(0\Delta 0)} + V_\beta^{(1\Delta 1)} + V_\beta^{(2\Delta 2)} + \dots \quad (33)$$

In particular, the first order magnetic part reads

$$V_{\text{magn}}^{(\partial)} = g \sqrt{\frac{5}{2}} C_{2A}^{1k}{}_{2B} \int d^3\mathbf{x} (\phi_1 \phi_2)_A^{(2)} i\partial_k (\phi_3)_B^{(2)}, \quad (34)$$

and the second order magnetic part is

$$V_{\text{magn}}^{(\Delta)} = -\frac{1}{3} \int d^3\mathbf{x} \left[(\phi_3)^{(0)} \Delta (\phi_3)^{(0)} + (\phi_3)_A^{(2)} \Delta (\phi_3)_A^{(2)} \right] = V_{\text{magn}}^{(0\Delta 0)} + V_{\text{magn}}^{(2\Delta 2)}. \quad (35)$$

From the second order term (35), we shall need in this work only the part $V_{\text{magn}}^{(0\Delta 0)}$ and the expression

$$\tilde{X}_M^{(S)} X_M^{(S)} \Big|_{\text{magn}} = \frac{1}{3} \left[(\phi_3)^{(0)} (\phi_3)^{(0)} + (\phi_3)_A^{(2)} (\phi_3)_A^{(2)} \right] = \frac{1}{\sqrt{3}} (\phi_3^2)^{(0)}. \quad (36)$$

The first order electric term consists of transitions from spin-0 and spin-2 to spin-1 fields and therefore does not contribute. Of the second order electric term I shall here only need ($\hat{\phi}_i \equiv \phi_j + \phi_k$, i, j, k cyclic)

$$V_{\text{elec}}^{(0\Delta 0)} = \frac{1}{3g^2} \int d^3\mathbf{x} \left[\left(\frac{1}{\hat{\phi}_3^2} \right)^{(0)} \Delta (T_3)^{(0)} + \frac{1}{4} \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} \Delta \left(\frac{1}{\hat{\phi}_1^3} \xi_1^2 + \frac{1}{\hat{\phi}_2^3} \xi_2^2 \right)^{(0)} \right. \\ \left. - \left(\frac{1}{\hat{\phi}_3^2} \tilde{\pi}_3 \right)^{(0)} \Delta (\tilde{\pi}_3)^{(0)} - \delta(\mathbf{0})^2 \left(\frac{1}{\hat{\phi}_3^2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} \Delta \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} \right], \quad (37)$$

with the functions (i, j, k cyclic)

$$T_i = \frac{1}{2} \pi_i^2 + i\delta(\mathbf{0}) \phi_i \left(\frac{1}{\phi_j^2 - \phi_i^2} + \frac{1}{\phi_k^2 - \phi_i^2} \right) \pi_i + \frac{1}{4} \frac{\phi_k^2 + \phi_i^2}{(\phi_k^2 - \phi_i^2)^2} \xi_j^2 + \frac{1}{4} \frac{\phi_j^2 + \phi_i^2}{(\phi_j^2 - \phi_i^2)^2} \xi_k^2, \\ \tilde{\pi}_i = \pi_i + i\delta(\mathbf{0}) \phi_i \left(\frac{1}{\phi_j^2 - \phi_i^2} + \frac{1}{\phi_k^2 - \phi_i^2} \right),$$

and

$$\tilde{X}_M^{(S)} X_M^{(S)}|_{\text{elec}} = \frac{1}{\sqrt{3} g^2} \left[\left(\frac{1}{\hat{\phi}_3^2} T_3 \right)^{(0)} + \frac{1}{4} \left(\frac{1}{\hat{\phi}_3} \left(\frac{1}{\hat{\phi}_1^3} \xi_1^2 + \frac{1}{\hat{\phi}_2^3} \xi_2^2 \right) \right)^{(0)} - \frac{1}{2} \frac{1}{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} (\xi_1^2 + \xi_2^2 + \xi_3^2) \right. \\ \left. + \frac{\delta(\mathbf{0})^2}{8} \left(21 \left(\frac{1}{\hat{\phi}_1^2 \hat{\phi}_2^2} \right)^{(0)} + 11 \left(\frac{1}{\hat{\phi}_3^3} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} + 3 \frac{1}{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} \right) \right]. \quad (38)$$

The first order measure part, containing spin-2 and spin-3 fields⁵, reads

$$V_{\text{meas}}^{(\partial)} = \frac{\delta(\mathbf{0})}{24g} \sqrt{\frac{5}{2}} C_{2A \ 2B}^{1k} \int d^3\mathbf{x} \left[\left[6 \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)_A^{(2)} + 3 \left(\frac{\hat{\phi}_3}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)_A^{(2)} - 3 \left(\frac{1}{\hat{\phi}_3} \left(\frac{\hat{\phi}_1}{\hat{\phi}_2^2} + \frac{\hat{\phi}_2}{\hat{\phi}_1^2} \right) \right)_A^{(2)} \right] i\partial_k \left(\frac{1}{\hat{\phi}_3} \right)_B^{(2)} \right. \\ \left. + \left[3 \left(\frac{\hat{\phi}_1}{\hat{\phi}_2^2} + \frac{\hat{\phi}_2}{\hat{\phi}_1^2} \right)_A^{(2)} + \left(\frac{1}{\hat{\phi}_3^2} \left(\frac{\hat{\phi}_1^2}{\hat{\phi}_2} + \frac{\hat{\phi}_2^2}{\hat{\phi}_1} \right) \right)_A^{(2)} - \left(\frac{1}{\hat{\phi}_3} \left(\frac{\hat{\phi}_1^2}{\hat{\phi}_2^2} + \frac{\hat{\phi}_2^2}{\hat{\phi}_1^2} \right) \right)_A^{(2)} \right] i\partial_k \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)_B^{(2)} \right] \\ + \frac{\delta(\mathbf{0})}{8g} \sqrt{\frac{7}{3}} C_{3M \ 2B}^{1k} \int d^3\mathbf{x} \left[3 \left(\frac{1}{\hat{\phi}_3^2} \cdot \hat{\phi}_3 \right)_M^{(3)} + \left(\frac{1}{\hat{\phi}_3} \left(\frac{1}{\hat{\phi}_1^2} + \frac{1}{\hat{\phi}_2^2} \right) \cdot \hat{\phi}_3^2 \right)_M^{(3)} \right] i\partial_k \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)_B^{(2)}, \quad (39)$$

Of the second order measure term, which is very complicated, I shall need here only

$$V_{\text{meas}}^{(0\Delta 0)} = \frac{\delta(\mathbf{0})^2}{48 g^2} \int d^3\mathbf{x} \left[\left[-9 \left(\frac{1}{\hat{\phi}_3^2} \right)^{(0)} + 8 \left(\frac{\hat{\phi}_3}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} - \left(\frac{\hat{\phi}_3^2}{\hat{\phi}_1^2 \hat{\phi}_2^2} \right)^{(0)} \right] \Delta \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)^{(0)} \right. \\ + \frac{1}{5} \left[14 \left(\frac{1}{\hat{\phi}_3^2} \right)^{(0)} + 42 \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)^{(0)} - 5 \left(\frac{\hat{\phi}_3}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} + 4 \left(\frac{\hat{\phi}_3^2}{\hat{\phi}_1^2 \hat{\phi}_2^2} \right)^{(0)} \right] \Delta \left[\left(\frac{1}{\hat{\phi}_3^2} \right)^{(0)} - \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \right)^{(0)} \right] \\ - 4 \left[2 \left(\frac{1}{\hat{\phi}_3^3} \right)^{(0)} + \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \right)^{(0)} \right] \Delta \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} + \frac{2}{5} \left[2 \left(\frac{1}{\hat{\phi}_3^3} \right)^{(0)} - \left(\frac{\hat{\phi}_3}{\hat{\phi}_1^2 \hat{\phi}_2^2} \right)^{(0)} - 2 \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \right)^{(0)} \right. \\ \left. - \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} - \left(\frac{\hat{\phi}_3}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1^2} + \frac{1}{\hat{\phi}_2^2} \right) \right)^{(0)} \right] \Delta \left[\left(\frac{\hat{\phi}_3}{\hat{\phi}_1 \hat{\phi}_2} \right)^{(0)} - \left(\frac{1}{\hat{\phi}_3} \right)^{(0)} \right] \\ \left. + \left(\frac{1}{\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3} \right)^{(0)} \Delta \left[\left(\frac{\hat{\phi}_3^2}{\hat{\phi}_1 \hat{\phi}_2} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} - \left(\hat{\phi}_3 \left(\frac{1}{\hat{\phi}_1^2} + \frac{1}{\hat{\phi}_2^2} \right) \right)^{(0)} \right] \right], \quad (40)$$

and

$$\tilde{X}_M^{(S)} X_M^{(S)}|_{\text{meas}} = -\frac{\delta(\mathbf{0})^2}{8\sqrt{3} g^2} \int d^3\mathbf{x} \left[3 \left(\frac{1}{\hat{\phi}_1^2 \hat{\phi}_2^2} \right)^{(0)} + 6 \left(\frac{1}{\hat{\phi}_3^3} \left(\frac{1}{\hat{\phi}_1} + \frac{1}{\hat{\phi}_2} \right) \right)^{(0)} \right]. \quad (41)$$

⁵Using the notation $(\phi_3 \cdot \psi_3)_M^{(3)} := [(\phi_2 \psi_3 - \phi_3 \psi_2) + (\phi_3 \psi_1 - \phi_1 \psi_3) + (\phi_1 \psi_2 - \phi_2 \psi_1)] (D_{M \ 2}^{(3)}(\chi) - D_{M \ -2}^{(3)}(\chi)) / (2\sqrt{3})$

4 Coarse graining and strong coupling expansion in $\lambda = g^{-2/3}$

I now set an ultraviolet cutoff a by introducing an infinite spatial lattice of granulas $G(\mathbf{n}, a)$, here cubes of length a , situated at sites $\mathbf{x} = a\mathbf{n}$ ($\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$), and considering the averaged variables

$$\phi(\mathbf{n}) := \frac{1}{a^3} \int_{G(\mathbf{n}, a)} d\mathbf{x} \phi(\mathbf{x}) \quad (42)$$

(where in particular $\delta(\mathbf{0}) \rightarrow 1/a^3$), and the discretized first and second spatial derivatives ($s=1,2,3$),

$$\partial_s \phi(\mathbf{n}) := \lim_{N \rightarrow \infty} \sum_{n=1}^N w_N(n) \frac{1}{2na} (\phi(\mathbf{n} + n\mathbf{e}_s) - \phi(\mathbf{n} - n\mathbf{e}_s)) \quad (43)$$

$$\partial_s^2 \phi(\mathbf{n}) := \lim_{N \rightarrow \infty} \sum_{n=1}^N w_N(n) \frac{1}{(na)^2} (\phi(\mathbf{n} + n\mathbf{e}_s) + \phi(\mathbf{n} - n\mathbf{e}_s) - 2\phi(\mathbf{n})) \quad (44)$$

with the unit lattice vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ and the distribution

$$w_N(n) := 2 \frac{(-1)^{n+1} (N!)^2}{(N-n)!(N+n)!}, \quad 1 \leq n \leq N, \quad \sum_{n=1}^N w_N(n) = 1. \quad (45)$$

The values of $\partial_s \phi(\mathbf{n})$ and $\partial_s^2 \phi(\mathbf{n})$ in (43) and (44) for a given site \mathbf{n} and direction, say $s = 1$, are chosen to coincide with the first and second derivative, $I'_{2N}(an_1)|_{n_2, n_3}$ and $I''_{2N}(an_1)|_{n_2, n_3}$ ⁶, respectively, of the interpolation polynomial $I_{2N}(x_1)|_{n_2, n_3}$ in the x_1 coordinate, which is uniquely determined by the series of values $\phi(n_1 + n, n_2, n_3)$ ($n = -N, \dots, N$) obtained via the averaging (42), and then taking the limit $N \rightarrow \infty$. Note, that the ($N = 1$) choice, $\partial_s \phi(\mathbf{n})|_{N=1} = (\phi(\mathbf{n} + \mathbf{e}_s) - \phi(\mathbf{n} - \mathbf{e}_s)) / (2a)$ and $\partial_s^2 \phi(\mathbf{n})|_{N=1} = (\phi(\mathbf{n} + \mathbf{e}_s) + \phi(\mathbf{n} - \mathbf{e}_s) - 2\phi(\mathbf{n})) / a^2$, which includes only the nearest neighbors $\mathbf{n} \pm \mathbf{e}_s$, would lead to the same results as (43) and (44) for the soft components of the original field $\phi(\mathbf{x})$, which vary only slightly over several lattice sites, but lead to values falling off faster than (43) and (44) for higher momentum components approaching π/a .

Applying furthermore the rescaling transformation (afterwards again dropping the primes)

$$\phi_i = \frac{g^{-1/3}}{a} \phi'_i, \quad \pi_i = \frac{g^{1/3}}{a^2} \pi'_i, \quad \xi_i = \frac{1}{a^3} \xi'_i, \quad (46)$$

I obtain the expansion of the Hamiltonian in $\lambda = g^{-2/3}$

$$H = \frac{g^{2/3}}{a} \left[\mathcal{H}_0 + \lambda \sum_{\alpha} \mathcal{V}_{\alpha}^{(\partial)} + \lambda^2 \left(\sum_{\beta} \mathcal{V}_{\beta}^{(\Delta)} + \sum_{\gamma} \mathcal{V}_{\gamma}^{(\partial\partial \neq \Delta)} \right) + \mathcal{O}(\lambda^3) \right], \quad (47)$$

with the "free" Hamiltonian

$$\begin{aligned} \mathcal{H}_0 = \sum_{\mathbf{n}} \left[\frac{1}{2} \sum_{ijk}^{\text{cyclic}} \left[\pi_i^2(\mathbf{n}) - \frac{2i}{\phi_j^2(\mathbf{n}) - \phi_k^2(\mathbf{n})} (\phi_j(\mathbf{n})\pi_j(\mathbf{n}) - \phi_k(\mathbf{n})\pi_k(\mathbf{n})) \right. \right. \\ \left. \left. + \xi_i^2(\mathbf{n}) \frac{\phi_j^2(\mathbf{n}) + \phi_k^2(\mathbf{n})}{(\phi_j^2(\mathbf{n}) - \phi_k^2(\mathbf{n}))^2} + \phi_j^2(\mathbf{n})\phi_k^2(\mathbf{n}) \right] \right] = \sum_{\mathbf{n}} \mathcal{H}_0^{QM}(\mathbf{n}), \end{aligned} \quad (48)$$

which is the sum of the Hamiltonians of $SU(2)$ -Yang-Mills quantum mechanics of constant fields in each box, and the interaction parts, relating different boxes,

$$\mathcal{V}_{\alpha}^{(\partial)} = \lim_{N \rightarrow \infty} \sum_{n=1}^N w_N(n) \left[\frac{i}{2n} C_{S_1 M_1}^{1s}{}_{S_2 M_2} \sum_{\mathbf{n}} \tilde{\mathcal{Y}}_{\alpha M_1}^{(S_1)}[\phi(\mathbf{n})] \left(\mathcal{Y}_{\alpha M_2}^{(S_2)}[\phi(\mathbf{n} + n\mathbf{e}_s)] - \mathcal{Y}_{\alpha M_2}^{(S_2)}[\phi(\mathbf{n} - n\mathbf{e}_s)] \right) \right] \quad (49)$$

$$\mathcal{V}_{\beta}^{(\Delta)} = -\lim_{N \rightarrow \infty} \sum_{n=1}^N w_N(n) \left[\frac{1}{n^2} \sum_{\mathbf{n}, s} \tilde{\mathcal{X}}_{\beta M}^{(S)}[\phi(\mathbf{n})] \left(\mathcal{X}_{\beta M}^{(S)}[\phi(\mathbf{n} + n\mathbf{e}_s)] + \mathcal{X}_{\beta M}^{(S)}[\phi(\mathbf{n} - n\mathbf{e}_s)] - 2\mathcal{X}_{\beta M}^{(S)}[\phi(\mathbf{n})] \right) \right] \quad (50)$$

⁶Differentiating the Lagrange interpolation polynomials $I_{2N}(x)$ with given values y_n at the equidistant points $x_n = x_0 + na$, ($n = -N, -N+1, \dots, N-1, N$), once/twice at the central point x_0 , one obtains: $I'_{2N}(x_0) = \sum_{n=1}^N w_N(n)(y_n - y_{-n})/(2na)$ and $I''_{2N}(x_0) = \sum_{n=1}^N w_N(n)(y_n + y_{-n} - 2y_0)/(na)^2$ with the distribution (45). For $N=1$, in particular, one has $I'_2(x_0) = (y_1 - y_{-1})/(2a)$ and $I''_2(x_0) = (y_1 + y_{-1} - 2y_0)/a^2$

with the dimensionless and coupling constant independent terms \mathcal{X}, \mathcal{Y} , obtained from the X, Y in (32) and (33) by putting $\mathcal{X}[\phi] := X[\phi]|_{a=1, g=1, \delta(0)=1}$ and $\mathcal{Y}[\phi] := Y[\phi]|_{a=1, g=1, \delta(0)=1}$.

The expansion of the Hamiltonian in terms of the number of spatial derivatives is therefore equivalent to a strong coupling expansion in $\lambda = g^{-2/3}$. It is the analogon of the weak coupling expansion in $g^{2/3}$ for small boxes by Lüscher and Münster[4],[5]⁷, and supplies a useful alternative to strong coupling expansions based on the Wilson-loop gauge invariant variables, which had been carried out by Kogut, Sinclair, and Susskind [6] for a 3-dimensional spatial lattice in the Hamiltonian formalism, yielding an expansion in $1/g^4$, and by Münster [7] for a 4-dimensional space-time lattice.

The low energy spectrum and eigenstates of \mathcal{H}_0^{QM} at each site \mathbf{n} appearing in (48),

$$\mathcal{H}_0^{QM}(\mathbf{n})|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} = \epsilon_i^{(S)}(\mathbf{n})|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}}, \quad (51)$$

characterised by the quantum numbers of spin S, M , have been obtained in [5],[8],[9] with high accuracy. It is important to note (see [9] for details), that at strong coupling, due to the positivity of the range (23), all states should satisfy either the (+) b.c. $\partial_\phi \Phi(\phi)|_{\phi=0} = 0$, or the (-) b.c. $\Phi(\phi)|_{\phi=0} = 0$, in accordance with (23) and the invariance of the Hamiltonian \mathcal{H}_0 under parity transformation $\phi \rightarrow -\phi$. The spectrum is purely discrete in both cases and the lowest energies are

$$\epsilon_0^+ = 4.1167, \quad \epsilon_0^- = 8.7867. \quad (52)$$

The energies (relative to ϵ_0)

$$\mu_i^{(S)+} := \epsilon_i^{(S)+} - \epsilon_0^+, \quad \mu_i^{(S)-} := \epsilon_i^{(S)-} - \epsilon_0^-, \quad (53)$$

of the lowest states for spin-0,2,3 and 4 for (+) and (-) b.c. are summarized in Table 1a and 1b. Spin-1 states are absent for both cases. The underlined values correspond to stable excitations below threshold

$$\mu_{\text{th}}^+ = 3.796 \quad (= 2\mu_1^{(2)+}), \quad \mu_{\text{th}}^- = 5.089 \quad (= 2\mu_1^{(2)-}), \quad (54)$$

for decay into two spin-2 excitations $\mu_1^{(2)}$ (lightest in the spectrum).

$\mu_i^{(S)+}$	$S=0$	$S=2$	$S=3$	$S=4$	$\mu_i^{(S)-}$	$S=0$	$S=2$	$S=3$	$S=4$
$i=1$	<u>2.270</u>	<u>1.898</u>	8.009	<u>3.61</u>	$i=1$	<u>3.268</u>	<u>2.545</u>	9.250	<u>4.93</u>
$i=2$	3.857	<u>3.704</u>	10.815	5.23	$i=2$	5.233	5.212	12.78	7.37
$i=3$	5.09	5.22	13.1	6.9	$i=3$	6.803	6.612	15.38	9.6

Table 1a and 1b: Results for the first three excitation energies $\mu_i^{(S)}$ for (+) and (-) b.c. The underlined values correspond to stable excitations below threshold (54). The numerical errors (estimated from the deviation from the virial theorem, see [9]) are smaller than the last digit in the numbers given.

5 Perturbation theory in $\lambda = g^{-2/3}$

5.1 Free many-glueball states

The eigenstates of the free Hamiltonian

$$H_0 = \frac{g^{2/3}}{a} \sum_{\mathbf{n}} \mathcal{H}_0^{QM}(\mathbf{n})$$

are free many-glueball states (completely decoupled granulas). The free glueball vacuum is

$$|0\rangle \equiv \bigotimes_{\mathbf{n}} |\Phi_0\rangle_{\mathbf{n}} \rightarrow E_{\text{vac}}^{\text{free}} = \mathcal{N} \epsilon_0 \frac{g^{2/3}}{a}$$

⁷ Integrating out all higher modes in a small box of size a , a weak coupling expansion for energies of the constant fields, $E = \frac{1}{a} \sum_{k=0}^{\infty} \epsilon_k \bar{\lambda}^k$, $\bar{\lambda} \equiv [\bar{g}(\Lambda_{MS} a)]^{2/3}$ is obtained, with the standard running coupling constant in the MS scheme.

(\mathcal{N} total number of granulas) with all granulas in the lowest state of energy ϵ_0 . The free one-glueball states, which in this work I choose to be momentum eigenstates, are

$$|S, M, i, \mathbf{k}\rangle \equiv \sum_{\mathbf{n}} e^{i\mathbf{a}\mathbf{k}\cdot\mathbf{n}} \left[|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m} \neq \mathbf{n}} |\Phi_0\rangle_{\mathbf{m}} \right] \rightarrow E_i^{(S)\text{free}}(k) = \mu_i^{(S)} \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}},$$

the free two-glueball states,

$$\begin{aligned} |(S_1, M_1, i_1, \mathbf{n}_1), (S_2, M_2, i_2, \mathbf{n}_2)\rangle &\equiv |\Phi_{i_1, M_1}^{(S_1)}\rangle_{\mathbf{n}_1} \otimes |\Phi_{i_2, M_2}^{(S_2)}\rangle_{\mathbf{n}_2} \left[\bigotimes_{\mathbf{m} \neq \mathbf{n}_1, \mathbf{n}_2} |\Phi_0\rangle_{\mathbf{m}} \right] \\ &\rightarrow E_{i_1, i_2}^{(S_1, S_2)\text{free}} = (\mu_{i_1}^{(S_1)} + \mu_{i_2}^{(S_2)}) \frac{g^{2/3}}{a} + E_{\text{vac}}^{\text{free}}, \end{aligned}$$

and so on. Matrix elements between these free glueball states are calculated using the measure (31).

5.2 Interacting glueball vacuum

The energy of the interacting glueball vacuum up to λ^2

$$\begin{aligned} E_{\text{vac}} = \mathcal{N} \frac{g^{2/3}}{a} \left[\epsilon_0 + \lambda^2 \sum_{\beta} \langle 0 | \mathcal{V}_{\beta}^{(\Delta)} | 0 \rangle - \lambda^2 \left(\sum_{\alpha, \alpha'} \sum_{i_1, i_2} \frac{\langle 0 | \mathcal{V}_{\alpha, 2-2}^{(\partial)} | 2_{i_1} 2_{i_2} \rangle \langle 2_{i_1} 2_{i_2} | \mathcal{V}_{\alpha', 2-2}^{(\partial)} | 0 \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)}} \right. \right. \\ \left. \left. + \sum_{i_1, i_2} \frac{\langle 0 | \mathcal{V}_{2-3}^{(\partial)} | 2_{i_1} 3_{i_2} \rangle \langle 2_{i_1} 3_{i_2} | \mathcal{V}_{2-3}'^{(\partial)} | 0 \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(3)}} \right) + \mathcal{O}(\lambda^3) \right] \equiv \frac{g^{2/3}}{a} \left[\epsilon_0 + c_0 \lambda^2 + \mathcal{O}(\lambda^3) \right] \quad (55) \end{aligned}$$

is obtained using first and second order perturbation theory.

For any $\mathcal{V}_{\beta}^{(\Delta)}$ of (50) I obtain, using $\lim_{N \rightarrow \infty} \sum_{n=1}^N (w_N(n)/n^2) = \zeta(2) = \pi^2/6$,

$$c_0 \Big|_{\beta}^{\text{1st ord}} = \langle 0 | \mathcal{V}_{\beta}^{(\Delta)} | 0 \rangle = \pi^2 \left[\langle \Phi_0 | \left(\tilde{\mathcal{X}}_{\beta, M}^{(S)} \mathcal{X}_{\beta, M}^{(S)} \right) | \Phi_0 \rangle - \langle \Phi_0 | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_0 \rangle \langle \Phi_0 | \mathcal{X}_{\beta}^{(0)} | \Phi_0 \rangle \right]. \quad (56)$$

For example, for the magnetic potential $\mathcal{V}_{\text{magn}}^{(\Delta)}$, corresponding to the $V_{\text{magn}}^{(\Delta)}$ in (35), Eq. (56) becomes

$$c_0 \Big|_{\text{magn}}^{\text{1st ord}} = \frac{\pi^2}{3} \left[\langle \Phi_0 | \phi_1^2 + \phi_2^2 + \phi_3^2 | \Phi_0 \rangle - \frac{1}{3} \langle \Phi_0 | \phi_1 + \phi_2 + \phi_3 | \Phi_0 \rangle^2 \right] = 4.560 \text{ (3.514)}.$$

Here (and in the following paragraphs) the number without brackets corresponds to the (+) b.c. and the number in brackets to the (−) b.c. The numerical errors are smaller than the last digit in the numbers given. Together with the corresponding contributions 42.323 (13.229) for the electric $\mathcal{V}_{\text{elec}}^{(\Delta)}$ and −16.408 (−1.782) for the measure terms $\mathcal{V}_{\text{meas}}^{(\Delta)}$, I find the total first order

$$c_0 \Big|_{\text{tot}}^{\text{1st ord}} = 30.474 \text{ (14.962)}.$$

Using $\mathcal{V}_{\alpha}^{(\partial)}$ in (49) and $\lim_{N \rightarrow \infty} \sum_{n=1}^N (w_N^2(n)/n^2) = 4\zeta(2) = 2\pi^2/3$, I obtain for the contribution due to the vacuum polarization into a virtual pair of spin-2 particles,

$$\begin{aligned} c_0 \Big|_{2-2, \alpha, \alpha'}^{\text{2nd ord}} &= -\frac{\pi^2}{25} \sum_{i_1, i_2} \frac{\langle \Phi_0 | \tilde{\mathcal{Y}}_{\alpha}^{(2)} | \Phi_{i_1}^{(2)} \rangle \langle \Phi_0 | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_{i_2}^{(2)} \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)}} \left[\langle \Phi_{i_1}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle \right. \\ &\quad \left. + \langle \Phi_{i_2}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_1}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle \right], \quad (57) \end{aligned}$$

and similarly that due to the vacuum polarization into a spin-2 and a spin-3 particle,

$$c_0 \Big|_{2-3}^{\text{2nd ord}} = -\frac{\pi^2}{35} \sum_{i_1, i_2} \frac{\langle \Phi_0 | \tilde{\mathcal{Y}}^{(3)} | \Phi_{i_1}^{(3)} \rangle \langle \Phi_0 | \mathcal{Y}^{(2)} | \Phi_{i_2}^{(2)} \rangle}{\mu_{i_1}^{(3)} + \mu_{i_2}^{(2)}} \langle \Phi_{i_1}^{(3)} | \tilde{\mathcal{Y}}'^{(3)} | \Phi_0 \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}^{(2)} | \Phi_0 \rangle.$$

The leading contribution to (57) comes from the $\mathcal{V}_{\text{magn}}^{(\partial)} - \mathcal{V}_{\text{magn}}^{(\partial)}$ vacuum polarization (see $V_{\text{magn}}^{(\partial)}$ in (34))

$$c_0 \Big|_{\text{magn-magn}}^{\text{2nd ord}} = -\frac{\pi^2}{10} \sum_{i_1, i_2} \frac{\langle \Phi_0 | |(\phi_1 \phi_2)^{(2)}| | \Phi_{i_1}^{(2)} \rangle \langle \Phi_0 | |(\phi_3)^{(2)}| | \Phi_{i_2}^{(2)} \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)}} \left[\langle \Phi_{i_1}^{(2)} | |(\phi_1 \phi_2)^{(2)}| | \Phi_0 \rangle \langle \Phi_{i_2}^{(2)} | |(\phi_3)^{(2)}| | \Phi_0 \rangle \right. \\ \left. + \langle \Phi_{i_2}^{(2)} | |(\phi_1 \phi_2)^{(2)}| | \Phi_0 \rangle \langle \Phi_{i_1}^{(2)} | |(\phi_3)^{(2)}| | \Phi_0 \rangle \right] = -0.399 \ (-0.341) .$$

Together with the smaller contributions -0.1516 (-0.0186) from $\mathcal{V}_{\text{magn}}^{(\partial)} - \mathcal{V}_{\text{meas}}^{(\partial)}$, -0.0295 (-0.0004) from $\mathcal{V}_{\text{meas}}^{(\partial)} - \mathcal{V}_{\text{meas}}^{(\partial)}$, and the negligibly small -3.5×10^{-5} (-8.6×10^{-5}) from $\mathcal{V}_{2-3}^{(\partial)} - \mathcal{V}_{2-3}^{(\partial)}$, I find the total second order

$$c_0 \Big|_{\text{tot}}^{\text{2nd ord}} = -0.580 \ (-0.360) .$$

Hence 1st and 2nd order perturbation theory together give the result

$$E_{\text{vac}}^+ = \mathcal{N} \frac{g^{2/3}}{a} \left[4.1167 + 29.894\lambda^2 + \mathcal{O}(\lambda^3) \right] , \quad E_{\text{vac}}^- = \mathcal{N} \frac{g^{2/3}}{a} \left[8.7867 + 14.602\lambda^2 + \mathcal{O}(\lambda^3) \right] , \quad (58)$$

for the energy of the interacting glueball vacuum up to λ^2 , for the (+) and (-) boundary conditions, respectively. The results are summarized in Table 2.

vacuum	ϵ_0	$c_0^{(1\text{st})}$	$c_0^{(2\text{nd})}$	c_0	c_0/ϵ_0
(+)	4.1167	30.474	-0.580	29.894	7.262
(-)	8.7867	14.962	-0.360	14.602	1.662

Table 2: Results for the interacting glueball vacuum for (+) and (-) b.c. The numerical errors are smaller than the last digits in the numbers shown.

5.3 Interacting Spin-0 glueballs

Including interactions V^Δ and V^∂ using 1st and 2nd order perturbation theory, we obtain the following energy of the interacting spin-0 glueball up to λ^2 ,

$$E_i^{(0)}(k) - E_{\text{vac}} = \frac{g^{2/3}}{a} \left[\mu_i^{(0)} + \lambda^2 \sum_{\beta} \langle 0ik | \mathcal{V}_{\beta}^{(\Delta)} | 0ik \rangle - \lambda^2 \left(\sum_{\alpha, \alpha'} \sum_{i_1, i_2} \frac{\langle 0ik | \mathcal{V}_{\alpha, 2-2}^{(\partial)} | 2_{i_1} 2_{i_2} \rangle \langle 2_{i_1} 2_{i_2} | \mathcal{V}_{\alpha', 2-2}^{(\partial)} | 0ik \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)} - \mu_i^{(0)}} \right. \right. \\ \left. \left. + \sum_{i_1, i_2} \frac{\langle 0ik | \mathcal{V}_{2-3}^{(\partial)} | 2_{i_1} 3_{i_2} \rangle \langle 2_{i_1} 3_{i_2} | \mathcal{V}_{2-3}'^{(\partial)} | 0ik \rangle}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(3)} - \mu_i^{(0)}} \right) + \mathcal{O}(\lambda^3) \right] \\ \equiv \frac{g^{2/3}}{a} \left[\mu_i^{(0)} + \lambda^2 \left(c_i^{(0)} + \tilde{c}_i^{(0)} a^2 k^2 + \mathcal{O}((a^2 k^2)^2) \right) + \mathcal{O}(\lambda^3) \right] \quad (59)$$

All spin-0 glueball excitations are unstable at tree-level, except for the lowest $\mu_1^{(0)}$, which is below threshold (54) for decay into two spin-2 glueballs.

For a potential term of the general form $\mathcal{V}_{\beta}^{(\Delta)}$ of (50), I find in first order perturbation theory

$$c_1^{(0)} \Big|_{\beta}^{\text{1st ord}} = \pi^2 \left[\langle \Phi_1^{(0)} | \left(\tilde{\mathcal{X}}_{\beta, M}^{(S)} \mathcal{X}_{\beta, M}^{(S)} \right) | \Phi_1^{(0)} \rangle - \langle \Phi_0 | \left(\tilde{\mathcal{X}}_{\beta, M}^{(S)} \mathcal{X}_{\beta, M}^{(S)} \right) | \Phi_0 \rangle \right. \\ \left. - \left(\langle \Phi_1^{(0)} | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle - \langle \Phi_0 | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_0 \rangle \right) \langle \Phi_0 | \mathcal{X}_{\beta}^{(0)} | \Phi_0 \rangle - \langle \Phi_0 | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_0 \rangle \left(\langle \Phi_1^{(0)} | \mathcal{X}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle - \langle \Phi_0 | \mathcal{X}_{\beta}^{(0)} | \Phi_0 \rangle \right) \right. \\ \left. - \langle \Phi_1^{(0)} | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_0 \rangle \langle \Phi_0 | \mathcal{X}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle - \langle \Phi_0 | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle \langle \Phi_1^{(0)} | \mathcal{X}_{\beta}^{(0)} | \Phi_0 \rangle \right] , \quad (60)$$

$$c_1^{(0)} \Big|_{\beta}^{\text{1st ord}} = \left[\langle \Phi_1^{(0)} | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_0 \rangle \langle \Phi_0 | \mathcal{X}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle + \langle \Phi_0 | \tilde{\mathcal{X}}_{\beta}^{(0)} | \Phi_1^{(0)} \rangle \langle \Phi_1^{(0)} | \mathcal{X}_{\beta}^{(0)} | \Phi_0 \rangle \right] . \quad (61)$$

For example, for the magnetic potential $\mathcal{V}_{\text{magn}}^{(\Delta)}$ corresponding to the $V_{\text{magn}}^{(\Delta)}$ in (35), Eqs. (60) and (61) give

$$\begin{aligned} c_1^{(0)} \Big|_{\text{magn}}^{\text{1st ord}} &= \frac{\pi^2}{3} \left[\left(\langle \Phi_1^{(0)} | \phi_1^2 + \phi_2^2 + \phi_3^2 | \Phi_i^{(0)} \rangle - \langle \Phi_0 | \phi_1^2 + \phi_2^2 + \phi_3^2 | \Phi_0 \rangle \right) - \frac{2}{9} |\langle \Phi_0 | \phi_1 + \phi_2 + \phi_3 | \Phi_1^{(0)} \rangle|^2 \right. \\ &\quad \left. - 2/3 \left(\langle \Phi_1^{(0)} | \phi_1 + \phi_2 + \phi_3 | \Phi_1^{(0)} \rangle \langle \Phi_0 | \phi_1 + \phi_2 + \phi_3 | \Phi_0 \rangle - \langle \Phi_0 | \phi_1 + \phi_2 + \phi_3 | \Phi_0 \rangle^2 \right) \right] \\ &= 5.296 \ (2.710) , \\ \tilde{c}_1^{(0)} \Big|_{\text{magn}}^{\text{1st ord}} &= \frac{2}{9} |\langle \Phi_0 | \phi_1 + \phi_2 + \phi_3 | \Phi_1^{(0)} \rangle|^2 = 0.050 \ (0.048) . \end{aligned}$$

Together with the electric contributions $c_1 = 14.161 \ (4.660)$, $\tilde{c}_1 = 0.3977 \ (0.1778)$ and the measure contributions $c_1 = -1.813 \ (-0.3597)$, $\tilde{c}_1 = 0.0119 \ (0.0016)$, I obtain the total value from first order perturbation theory

$$c_1^{(0)} \Big|_{\text{tot}}^{\text{1st ord}} = 17.643 \ (7.011) , \quad \tilde{c}_1^{(0)} \Big|_{\text{tot}}^{\text{1st ord}} = 0.460 \ (0.228) .$$

Furthermore, second order perturbation theory leads to a change in the mass to its virtual decay into two spin-2 particles or into one spin-2 and one spin-3 particle. Using $\mathcal{V}_\alpha^{(\partial)}$ in (49) I obtain for the case of the virtual decay into two spin-2 particles,

$$\begin{aligned} c_1^{(0)} \Big|_{2-2,\alpha,\alpha'}^{\text{2nd ord}} &= -\frac{\pi^2}{25} \sum_{i_1, i_2} \frac{1}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)} - \mu_1^{(0)}} \left[\langle \Phi_0 | \tilde{\mathcal{Y}}_\alpha^{(2)} | \Phi_{i_1}^{(2)} \rangle \langle \Phi_1^{(0)} | \mathcal{Y}_\alpha^{(2)} | \Phi_{i_2}^{(2)} \rangle + \langle \Phi_1^{(0)} | \tilde{\mathcal{Y}}_\alpha^{(2)} | \Phi_{i_1}^{(2)} \rangle \langle \Phi_0 | \mathcal{Y}_\alpha^{(2)} | \Phi_{i_2}^{(2)} \rangle \right] \times \\ &\quad \times \left[\langle \Phi_{i_1}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle + \langle \Phi_{i_1}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle \right. \\ &\quad \left. + \langle \Phi_{i_2}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_1}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle + \langle \Phi_{i_2}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \langle \Phi_{i_1}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle \right] , \quad (62) \end{aligned}$$

$$\begin{aligned} \tilde{c}_1^{(0)} \Big|_{2-2,\alpha,\alpha'}^{\text{2nd ord}} &= \frac{1}{100} \sum_{i_1, i_2} \frac{1}{\mu_{i_1}^{(2)} + \mu_{i_2}^{(2)} - \mu_1^{(0)}} \left[\langle \Phi_0 | \tilde{\mathcal{Y}}_\alpha^{(2)} | \Phi_{i_1}^{(2)} \rangle \langle \Phi_1^{(0)} | \mathcal{Y}_\alpha^{(2)} | \Phi_{i_2}^{(2)} \rangle \times \right. \\ &\quad \times \left(\langle \Phi_{i_1}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle + \langle \Phi_{i_2}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_1}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \right) \\ &\quad + \langle \Phi_1^{(0)} | \tilde{\mathcal{Y}}_\alpha^{(2)} | \Phi_{i_1}^{(2)} \rangle \langle \Phi_0 | \mathcal{Y}_\alpha^{(2)} | \Phi_{i_2}^{(2)} \rangle \left(\langle \Phi_{i_2}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \langle \Phi_{i_1}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_0 \rangle \right. \\ &\quad \left. + \langle \Phi_{i_1}^{(2)} | \tilde{\mathcal{Y}}_{\alpha'}^{(2)} | \Phi_0 \rangle \langle \Phi_{i_2}^{(2)} | \mathcal{Y}_{\alpha'}^{(2)} | \Phi_1^{(0)} \rangle \right) \Big] , \quad (63) \end{aligned}$$

and similar expressions for the virtual decay into one spin-2 and one spin-3 particle. The leading contribution to (62) and (63) comes from $\mathcal{V}_{\text{magn}}^{(\partial)} - \mathcal{V}_{\text{magn}}^{(\partial)}$ with $c_1 = -3.019 \ (-2.5238)$, $\tilde{c}_1 = 0.0226 \ (0.0194)$. Together with the smaller contributions $c_1 = -0.9947 \ (-1.001)$, $\tilde{c}_1 = 0.0055 \ (0.0005)$ from $\mathcal{V}_{\text{magn}}^{(\partial)} - \mathcal{V}_{\text{meas}}^{(\partial)}$, the contributions $c_1 = -0.1189 \ (-0.0013)$, $\tilde{c}_1 = 0.0036 \ (3 \times 10^{-6})$ from $\mathcal{V}_{\text{meas}}^{(\partial)} - \mathcal{V}_{\text{meas}}^{(\partial)}$, and the negligibly small contributions $c_1 = -8.3 \times 10^{-5} \ (-2.4 \times 10^{-4})$, $\tilde{c}_1 = 1.2 \times 10^{-7} \ (2.3 \times 10^{-7})$ from $\mathcal{V}_{2-3}^{(\partial)} - \mathcal{V}_{2-3}^{(\partial)}$, I obtain the total value from second order perturbation theory

$$c_1^{(0)} \Big|_{\text{tot}}^{\text{2nd ord}} = -4.133 \ (-2.626) , \quad \tilde{c}_1^{(0)} \Big|_{\text{tot}}^{\text{2nd ord}} = 0.028 \ (0.020) .$$

First and second order perturbation theory give the results (up to λ^2)

$$E_1^{(0)+}(k) - E_{\text{vac}}^+ = \left[2.270 + 13.510\lambda^2 + \mathcal{O}(\lambda^3) \right] \frac{g^{2/3}}{a} + 0.488 \frac{a}{g^{2/3}} k^2 + \mathcal{O}((a^2 k^2)^2) , \quad (64)$$

$$E_1^{(0)-}(k) - E_{\text{vac}}^- = \left[3.268 + 4.385\lambda^2 + \mathcal{O}(\lambda^3) \right] \frac{g^{2/3}}{a} + 0.248 \frac{a}{g^{2/3}} k^2 + \mathcal{O}((a^2 k^2)^2) . \quad (65)$$

for the energy spectrum of the interacting spin-0 glueball for the (+) and (-) boundary conditions, respectively. The results are summarized in Table 3⁸.

spin - 0	μ_1	$c_1^{(1st)}$	$c_1^{(2nd)}$	c_1	c_1/μ_1	$\tilde{c}_1^{(1st)}$	$\tilde{c}_1^{(2nd)}$	\tilde{c}_1	$1/(2\mu_1)$
(+)	2.270	17.643	-4.133	13.510	5.953	0.460	0.028	0.488	0.220
(-)	3.268	7.011	-2.626	4.385	1.342	0.228	0.020	0.248	0.153

Table 3: Results for the interacting spin-0 glueball for(+) and (-) b.c. The numerical errors are smaller than the last digits in the numbers shown.

5.4 Discussion of the results

First I would like to comment on the relation between the glueball mass and coupling constant renormalisation in the IR. Consider the physical mass

$$M = \frac{g_0^{2/3}}{a} \left[\mu + c g_0^{-4/3} \right] . \quad (66)$$

Demanding its independence of box size a , one obtains

$$\gamma(g_0) \equiv a \frac{d}{da} g_0(a) = \frac{3}{2} g_0 \frac{\mu + c g_0^{-4/3}}{\mu - c g_0^{-4/3}}$$

which vanishes for the two cases, $g_0 = 0$ or $g_0^{4/3} = -c/\mu$. The first solution corresponds to the perturbative fixed point, and the second, if it exists ($c < 0$), to an infrared fixed point. My result for $c_1^{(0)}/\mu_1^{(0)} = 5.95(1.34)$ suggests, that no infrared fixed points exist, in accordance with the corresponding result of Wilsonian lattice QCD⁹. Solving the above equation (66) for positive ($c > 0$) I obtain

$$g_0^{2/3}(Ma) = \frac{Ma}{2\mu} + \sqrt{\left(\frac{Ma}{2\mu}\right)^2 - \frac{c}{\mu}} , \quad a > a_c := 2\sqrt{c\mu}/M \quad (67)$$

with the physical glueball mass M . The minimal lattice sizes are $Ma_c = 11.08$ (7.57), corresponding to a critical coupling $g_0^2|_c = 14.52$ (1.55). Taking a typical physical glueball mass of $M \sim 1.6$ GeV [11], I obtain

$$\text{for } M \sim 1.6 \text{ GeV : } a_c \sim 1.4 \text{ fm (0.9 fm)} ,$$

which seems reasonable. The dependence of the results on the boundary conditions imposed, (+) or (-), might be seen as a prescription dependence. Of course, it will be much more effective to consider mass ratios, as soon as, for example, the spin-2 glueball will be calculated. It would also be interesting, to connect the behaviour of the glueball spectrum and the bare coupling constant (67), obtained for boxes of large size a , with those obtained for small boxes (see [4],[5] and footnote 7), in order to get information about the intermediate region, including the possibility of the occurrence of phase transitions.

Furthermore, I would like to remark, that Lorentz invariance imposes the following condition on the coefficients $\tilde{c}_i^{(S)}$ in (59):

$$E = \sqrt{m^2 + k^2} \simeq m + \frac{1}{2m} k^2 \quad \rightarrow \quad \tilde{c}_i^{(S)} = 1/(2\mu_i^{(S)}) .$$

Comparison of the last two columns of Table 3 show that my result does not satisfy this requirement by a factor of about 2. Of course, the glueball excitation carrying non-relativistic spin-0 considered here

⁸I would like to comment here, that, using only nearest neighbor interactions ($N = 1$ in (43) and (44) instead of $N \rightarrow \infty$), it would lead to the same \tilde{c}_1 , but a $(\pi^2/6) \simeq 1.64$ times smaller $c_1^{(1st)}|_{N=1} = 10.726(4.262)$ and a $(2\pi^2/3) \simeq 6.58$ times smaller $c_1^{(2nd)}|_{N=1} = -0.628(-0.399)$ and hence to a 25%(12%) smaller $c_1|_{N=1} = 10.097(3.863)$. Similarly for the vacuum (Table 2), it would lead to a 38% smaller $c_0|_{N=1} = 18.438(9.041)$.

⁹In comparison, the $SU(2)$ result from strong coupling on the lattice [7],[10]: $aM = 4 \log(g_0^2) + O(g_0^{-2}) \rightarrow \gamma(g_0) = \frac{1}{2} g_0 \log(g_0^2) + \dots$ does not contain infrared fixed points.

as a first step, does not correspond to a relativistic particle. Hence states of total angular momentum $J = S + L$, containing spin and orbital angular momentum, similar to the quark states in the Dirac wave-function, should be considered, e.g. the $J = 0$ state (using spherically symmetric granulation)

$$|J = 0, k\rangle \sim \alpha_1^{(0)} \sum_{\mathbf{n}} j_0(kr) \left[|\Phi_1^{(0)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m} \neq \mathbf{n}} |\Phi_0\rangle_{\mathbf{m}} \right] + \sum_{S,i}^{\text{stable}} \alpha_i^{(S)} \sum_{\mathbf{n}} j_S(kr) \sum_M Y_{SM}(\theta, \phi) \left[|\Phi_{i,M}^{(S)}\rangle_{\mathbf{n}} \bigotimes_{\mathbf{m} \neq \mathbf{n}} |\Phi_0\rangle_{\mathbf{m}} \right],$$

where the sum is over all excitations $\mu_i^{(S)} < \mu_{\text{th}}$, underlined in Table 1, which are stable at tree-level. For simplicity, I have considered in this work only the spin-0 excitation $\mu_1^{(0)+}$ ($\mu_1^{(0)-}$), but of course, also the lowest spin-2 excitations $\mu_1^{(2)+}$, $\mu_2^{(2)+}$ ($\mu_1^{(2)-}$) and the lowest spin-4 excitation $\mu_1^{(4)+}$ ($\mu_1^{(4)-}$) have to be included. Most important will certainly be the inclusion of the spin-2 state $\mu_1^{(2)+}$ ($\mu_1^{(2)-}$), which is lower in energy than the spin-0 state considered in this work. The necessary extension of the calculation to spin-2 and spin-4 states and the inclusion of orbital angular momentum of the lowest excitations clearly goes beyond the scope of this work.

6 Conclusions

It has been shown in this work, how a gauge invariant formulation of Yang-Mills theory on a 3-dimensional spatial lattice can be obtained by replacing integrals by sums and spatial derivatives by differences. This has been achieved by using the symmetric gauge $\epsilon_{ijk} A_{jk} = 0$ [2][3], and constructing the corresponding physical quantum Hamiltonian of $SU(2)$ Yang-Mills theory according to the general scheme given by Christ and Lee [1]. In contrast to the Coulomb gauge formulation, very suitable for the description of the high energy sector of the theory, the symmetric gauge quantum Hamiltonian, obtained here, is very suitable for the IR sector, since it can be expanded in the number of spatial derivatives. The "derivative-free" part of the Hamiltonian is just the sum of Hamiltonians of Yang-Mills quantum mechanics of constant fields for each granula (here a box of size a), with a purely discrete spectrum ("free glueballs"), and the terms of higher and higher number of spatial derivatives describing interactions between the constant fields of different granulas. This expansion has been carried out here explicitly and shown to be equivalent to a strong coupling expansion in $\lambda = g^{-2/3}$ for large box sizes a . It is the analogon to the weak coupling expansion in $g^{2/3}$ by Lüscher and Münster [4] [5], applicable for small boxes. Using the very accurate results of Yang-Mills quantum mechanics of constant fields in a box, obtained with the variational method in earlier work [9], the energy spectrum of weakly interacting glueballs can be calculated systematically and with high accuracy, using perturbation theory in λ . This offers a useful alternative to lattice calculations based on the Wilson-loop, including the corresponding analytic strong coupling expansions by Kogut, Sinclair, and Susskind [6] and Münster [7]. My result for the mass of the interacting spin-0 glueball up to λ^2 , as a first step, confirms the result of Wilsonian lattice QCD, that no infrared fixed points exist. Problems are a.o. the question of Lorentz invariance of the glueball spectrum.

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